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Hierarchical Bayesian modeling for uncertainty quantification and reliability updating using data

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Abstract

Quantifying uncertainty and updating reliability are essential for ensuring the safety and performance of engineering systems. This study develops a hierarchical Bayesian modeling (HBM) framework to quantify uncertainty and update reliability using data. By leveraging the probabilistic structure of HBM, the approach provides a robust solution for integrating model uncertainties and parameter variability into reliability assessments. The framework is applied to a linear mathematical model and a dynamical structural model. For the linear model, analytical solutions are derived for the hyper parameters and reliability, offering an efficient and precise means of uncertainty quantification and reliability evaluation. In the dynamical structural model, the posterior distributions of hyper parameters obtained from the HBM are used directly to update the reliability. This approach relies on the updated posteriors to reflect the influence of system uncertainties and dynamic behavior in the reliability predictions. The proposed approach demonstrates significant advantages over traditional Bayesian inference by addressing multi-source uncertainty in both static and dynamic contexts. This work highlights the versatility and computational efficiency of the HBM framework, establishing it as a powerful tool for uncertainty quantification and reliability updating in structural health monitoring and other engineering applications.

Keywords: uncertainty quantification, reliability updating, hierarchical Bayesian modeling, data-informed, structural dynamics

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Nomenclature

$\boldsymbol{\theta}$	Model parameters to be updated;
$\boldsymbol{\psi}$	Hyper parameter;
$\boldsymbol{\mu_\theta}$	The hyper mean;
$\boldsymbol{\Sigma_\theta}$	The hyper covariance matrix;
$\boldsymbol{D}$	Data set;
N_D	The number of datasets;
$\boldsymbol{A}$	A known system matrix;
σ^2	The variance of the measurement noise;
$\mathbf{I}$	Identity matrix;
$\boldsymbol{y}$	Observed data;
$p(\boldsymbol{\psi})$	The prior distribution of the hyper parameter;
$G(\boldsymbol{\theta})$	Limit state function;
b	Threshold value;
$\Phi(\cdot)$	Standard normal distribution;
$\boldsymbol{c}$	A weighting vector (or a direction vector);
β	The reliability index condition;
N_s	The number of samples for the model parameters;
$p(\boldsymbol{\theta}\|\boldsymbol{\psi})$	The joint Gaussian prior distribution of the model parameters;
$p(\boldsymbol{\theta},\boldsymbol{\psi}\|\boldsymbol{D})$	The joint distribution of the overall parameters;
$p(\boldsymbol{D}\|\boldsymbol{\theta},\boldsymbol{\psi})$	The joint likelihood function;
$\Pr(F\|\boldsymbol{D})$	Posterior failure probability;
$I_F(\boldsymbol{\theta},\boldsymbol{\varphi})$	Indicating function;
$p(\boldsymbol{\mu_\theta},\boldsymbol{\Sigma_\theta})$	The prior distribution of hyper parameters;
$p(\boldsymbol{\mu_\theta},\boldsymbol{\Sigma_\theta}\|\boldsymbol{D})$	Posteriori distribution of hyper parameters;
$p(\boldsymbol{D}\|\boldsymbol{\theta})$	The likelihood function;
$p(\boldsymbol{y}\|\boldsymbol{\theta})$	The likelihood function of linear model;
$F(\cdot)$	The failure probability conditional;
$\bar{\boldsymbol{\mu}}_{\boldsymbol{\theta}},\bar{\boldsymbol{\Sigma}}_{\boldsymbol{\theta}}$	The means of the hyper parameters;

1. Introduction

Uncertainty quantification and reliability updating play an essential role in the design, operation, and maintenance of engineering systems [1]. In real-world applications, uncertainties arise from various sources, including material properties, environmental and operational conditions, manufacturing variability, and modeling errors. These uncertainties can significantly impact the performance, safety, and longevity of engineering systems, making it crucial to quantify and account for such uncertainties [2, 3]. Reliability updating, in particular, involves leveraging new data to refine reliability predictions, ensuring they reflect the current state of the system [4]. This is particularly important for systems subjected to dynamic conditions, where operational demands and environmental factors may evolve over time. Accurate uncertainty quantification and reliability updating not only enhance decision-making but also improve the cost-efficiency of maintenance strategies and reduce the risk of failures in critical systems.

Bayesian inference has become a popular approach for addressing uncertainty in engineering analysis due to its strong theoretical foundation and flexibility [5–7]. It allows for the integration of prior knowledge-derived from historical data, expert judgment, or physical models-with newly observed data to infer parameters of interest. This probabilistic approach provides not just point estimates but also distributions that characterize uncertainty, enabling us to make informed decisions under uncertainty. By incorporating all sources of evidence in a mathematically consistent manner, Bayesian inference has been successfully applied across various domains, including structural health monitoring [8–10], prognostics [11–13], and risk assessment [14, 15]. In the context of reliability analysis, Bayesian methods naturally facilitate reliability updating as new information becomes available. Different strategies have been developed for reliability analysis. For example, the asymptotic approximation based on Laplace approximation [16], and the first- and second-order reliability methods [17] have been employed for reliability calculating. These approaches often provide computational efficiency when data is sufficient for reliability updating. In addition, the sampling approaches, which have been extensively explored in recent decades, have shown their advantages due to high accuracy in terms of the reliability calculations. Some relevant contributions on the sampling algorithms could be standard Monte Carlo algorithms [18, 19] and advanced sampling techniques including importance sampling [20–24], BUS algorithm (Bayesian updating with structural reliability methods) [25], subset simulations [26, 27] with Gibbs sampling [28], and other advanced sampling approaches [29–33]. Besides, the surrogate modeling, mainly Gaussian process, has been also explored for the context of reliability updating within the Bayesian perspective [34, 35]. Recently, a moment-based approach is also explored for reliability updating within the classical Bayesian framework [36]. It is worth noting that different likelihood functions provide advantages for specific problems. In addition to constructing likelihood functions based on differences between measurements and model predictions, various forms of uncertainty quantification metrics between model predictions and measurements, such as Bhattacharyya distance [37], Wasserstein distance metric [38], Jensen–Shannon divergence [39] and Bray-Curtis distance [40] have been developed. These metrics offer advantages in capturing different aspects of the discrepancy between distributions, such as sensitivity to shape differences, robustness to noise, and improved performance in high-dimensional spaces, thereby enhancing the flexibility and accuracy of model calibration and uncertainty quantification.

Despite its advantages, traditional Bayesian inference methods are not without limitations. A common issue arises from the underestimation of parameter uncertainties, especially when data appears in varying conditions or when overly simplistic prior distributions are used [41, 42]. For example, the datasets could be collected within different environmental conditions, or due to manufacturing or assembling process, or varied due to modeling error [43]. The estimates of the reliability using these datasets could vary because the uncertainty due to variability exists. The classical Bayesian approach may result in unreliable results if one combines the datasets and neglects such a variability. This is not due to the use of different sampling approaches but due to the fact that the parameter uncertainty would be decreased as the number of data increases [44]. This underestimation can result in

overconfident predictions, leading to reliability estimates that fail to account for the true variability and risks present in the system. Additionally, traditional Bayesian approaches sometimes rely on computational approximations, such as Laplace or variational approximations, which may introduce biases and further exacerbate the underestimation of uncertainty. These limitations highlight the need for more robust approaches that can better capture the complex and multi-source uncertainties inherent in engineering systems.

Hierarchical Bayesian modeling (HBM) provides a powerful extension to the traditional Bayesian framework, addressing the shortcomings of the traditional Bayesian framework [45] where the uncertainty due to variability is neglected as the number of datasets increases. The HBM framework has been developed in several scientific fields, such as structural dynamics [46–50], molecular dynamics [51–53], geotechnical engineering [54–56] to capture the uncertainty due to variability over the datasets. By introducing hierarchical structures, HBM models uncertainty at multiple levels, allowing for a more nuanced representation of variability arising from different sources, such as modeling errors, environmental influences, and manufacturing processes. This hierarchical structure enables HBM to explicitly distinguish between uncertainties associated with model parameters and those related to system variability, resulting in more realistic and reliable predictions. The flexibility of HBM supports its application to both linear and nonlinear systems, as well as static and dynamic problems.

Most existing HBM frameworks focus primarily on the quantification of parameter uncertainties, often neglecting the critical aspect of reliability updating. While these frameworks are effective in capturing uncertainties at multiple levels, they do not fully leverage the potential of HBM for updating reliability estimates as new data becomes available. This limitation restricts their applicability in dynamic environments where reliability assessments need to be continuously updated based on experimental observations. As a result, despite its potential the application of HBM for reliability updating remains underexplored, particularly in cases involving multiple datasets from either identical structures or a specific structure with different scenarios. These cases introduce uncertainty due to variability from dataset to dataset and such uncertainty cannot be ignored in reliability assessment, as it directly impacts the accuracy and robustness of predictions. This study is motivated by the need to address this gap and demonstrate the practical utility of HBM for uncertainty quantification and reliability updating in engineering applications.

This paper advances the state-of-the-art in uncertainty quantification and reliability updating by further developing the HBM framework. The study considers two distinct modeling scenarios: (1) a linear mathematical model and (2) a dynamical structural model. The HBM framework in both scenarios are consistent in fundamental principles, which provides a coherent and flexible probabilistic structure for uncertainty quantification and reliability updating. For the linear case, an analytical solution is derived for the hyper parameters and reliability index, showcasing computational efficiency. In contrast, the dynamical structural model presents challenges due to its complexity and the lack of an analytical solution. To address this, a two-stage sampling approach is employed, ensuring the accuracy of the results while maintaining computational feasibility. This work not only demonstrates the versatility of the HBM framework in handling different types of models but also highlights its ability to capture and quantify uncertainties comprehensively.

The structure of the paper is organized as follows. Section 2 provides an overview of the HBM framework, including its mathematical formulation and fundamental principles. Section 3 focuses on the linear mathematical model presenting the derivation of an analytical solution for uncertainty quantification and reliability, as well as a dynamical structural model describing the challenges associated with its complexity and the use of a two-stage sampling procedure to ensure accurate results. Section 4 presents two numerical examples to validate the proposed framework, demonstrating its effectiveness across both linear and dynamical cases. Section 5 concludes the paper by summarizing the key findings, discussing their implications, and suggesting potential directions for future research.

2. Brief review of the HBM framework

HBM is a statistical approach that can capture the uncertainty due to variability arising from modeling errors, environmental and operational conditions, as well as manufacturing and assembling processes. It is particularly useful for systems where uncertainty is inherent due to these sources of variability, and effective for dealing with multiple datasets and incorporating prior knowledge. The core of HBM lies in defining prior distributions for both model parameters and hyper parameters, followed by updating these priors with observed datasets to obtain posterior distributions. Consider a general model where the parameters $\boldsymbol{\theta} \in \mathbb{R}^{N_{\boldsymbol{\theta}}}$ are of interest, where $N_{\boldsymbol{\theta}}$ is the total number of unknown parameters in the set $\boldsymbol{\theta}$. In HBM, the model parameters typically follow a Gaussian prior distribution [57]:

$$p(\boldsymbol{\theta}|\psi) = N(\boldsymbol{\theta}|\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) \tag{1}$$

with hyper parameter ψ consisting of the hyper mean $\boldsymbol{\mu}_{\boldsymbol{\theta}} \in \mathbb{R}^{N_{\boldsymbol{\theta}}}$ and hyper covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\theta}} \in \mathbb{R}^{N_{\boldsymbol{\theta}} \times N_{\boldsymbol{\theta}}}$ to be identified from the available multiple datasets $\boldsymbol{D} = \{\boldsymbol{D}_i;\ i = 1, 2, \ldots, N_D\}$. Note that the Gaussian distribution is advocated based on the maximum entropy principle, which ensures that the chosen distribution makes the least informative assumptions given the available data. This makes it a natural choice for modeling uncertainties when limited information is available. Let $\boldsymbol{\theta}_i$ denote an independent realization of the parameter $\boldsymbol{\theta}$ from Gaussian distribution $N(\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}})$, corresponding to the ith dataset $\boldsymbol{D}_i$. The joint distribution of the overall parameters $p(\boldsymbol{\theta}, \psi|\boldsymbol{D})$ can be expressed [58] using the Bayes' theorem as:

$$p(\boldsymbol{\theta}, \psi|\boldsymbol{D}) \propto p(\boldsymbol{D}|\boldsymbol{\theta}, \psi)\, p(\boldsymbol{\theta}|\psi)\, p(\psi) \tag{2}$$

where $p(\boldsymbol{\psi})$ is the prior distribution of the hyper parameter, and $p(\boldsymbol{\theta}|\boldsymbol{\psi})$ and $p(\boldsymbol{D}|\boldsymbol{\theta},\boldsymbol{\psi})$ is the joint Gaussian prior distribution of the model parameters and the joint likelihood function. Given the independence between datasets, equation (2) is simplified as:

$$p(\boldsymbol{\theta},\boldsymbol{\psi}|\boldsymbol{D}) \propto p(\boldsymbol{\psi}) \prod_{i=1}^{N_D} p(\boldsymbol{\theta}_i|\boldsymbol{\psi}) p(\boldsymbol{D}_i|\boldsymbol{\theta}_i) \tag{3}$$

where $p(\boldsymbol{D}_i|\boldsymbol{\theta}_i)$ is the likelihood function that corresponds to the ith dataset $\boldsymbol{D}_i$.

Subsequently, the posterior distribution of the hyper parameter $p(\boldsymbol{\psi}|\boldsymbol{D})$ can be obtained by marginalizing the distribution $p(\boldsymbol{\theta},\boldsymbol{\psi}|\boldsymbol{D})$ over the model parameters $\boldsymbol{\theta}$. Two cases, including a linear mathematical model and a dynamical structure, will be discussed in the next section. In the first linear case, the property of the linear function will be utilized to derive an analytical solution for the posterior distribution of the model parameters, which corresponds to each dataset. This solution will be used to update the reliability of the system by incorporating the latest data. The analytical approach simplifies the process, offering an efficient means for reliability analysis, as the posterior distribution can be directly computed without the need for complex sampling methods. In the second case, a dynamical structure is considered, where the model exhibits more complexity due to the time-varying behavior and interdependencies among system parameters. Here, a sampling procedure will be employed to estimate the posterior distributions of the hyper parameters. This approach accommodates the non-linearity and uncertainty inherent in the dynamical system, ensuring that the reliability updates reflect both the variability of the system and the uncertainty from the load excitation. Through this combination of analytical and sampling methods, the proposed framework provides a versatile approach to uncertainty quantification and reliability updating. Details are given in the next section.

3. Uncertainty quantification and reliability updating using HBM

3.1. A linear mathematical model

The linear model considered in this study is expressed as:

$$g(\boldsymbol{\theta}) = \boldsymbol{A}^T\boldsymbol{\theta} \tag{4}$$

where $\boldsymbol{A}$ is a known system matrix, $\boldsymbol{\theta}$ is the model parameter vector, and the observed data $\boldsymbol{y}$ follows the likelihood function:

$$\boldsymbol{y} \sim N\left(\boldsymbol{A}^T\boldsymbol{\theta}, \sigma^2\mathbf{I}\right) \tag{5}$$

where σ^2 is the variance of the measurement noise. Expanding the likelihood function leads to the following form:

$$p(\boldsymbol{y}|\boldsymbol{\theta}) \propto \exp\left(-\frac{1}{2\sigma^2}\left(\boldsymbol{y}-\boldsymbol{A}^T\boldsymbol{\theta}\right)^T\left(\boldsymbol{y}-\boldsymbol{A}^T\boldsymbol{\theta}\right)\right). \tag{6}$$

Equation (6) can be also seen as a function of the model parameters $\boldsymbol{\theta}$, and can thus be written in a Gaussian form:

$$p(\boldsymbol{y}|\boldsymbol{\theta}) \propto N(\boldsymbol{\theta}|\boldsymbol{\theta}^*, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^*) \tag{7}$$

where the mean $\boldsymbol{\theta}^*$ and the covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^*$ are given by:

$$\begin{aligned} \boldsymbol{\theta}^* &= \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^*\boldsymbol{A}\sigma^{-2}\boldsymbol{y} \\ \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^* &= \sigma^2\left(\boldsymbol{A}\boldsymbol{A}^T\right)^{-1}. \end{aligned} \tag{8}$$

Assume that the datasets $\boldsymbol{D}$ are taken from the linear model with the ith dataset $\boldsymbol{D}_i$ equals to $\boldsymbol{y}_i$, the likelihood function $p(\boldsymbol{D}_i|\boldsymbol{\theta}_i)$ is then rewritten as:

$$p(\boldsymbol{D}_i|\boldsymbol{\theta}_i) \propto N\left(\boldsymbol{\theta}_i|\boldsymbol{\theta}_i^*, \boldsymbol{\Sigma}_{\boldsymbol{\theta}_i}^*\right) \tag{9}$$

where the mean $\boldsymbol{\theta}_i^*$ and the covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\theta}_i}^*$ for each dataset are calculated using the solutions in equation (8) by replacing by $\boldsymbol{\theta}^*$, $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^*$ and $\boldsymbol{y}$ by $\boldsymbol{\theta}_i^*$, $\boldsymbol{\Sigma}_{\boldsymbol{\theta}_i}^*$ and $\boldsymbol{y}_i$. respectively. It is notable that the variance of the measurement noise is assumed to be the same for each dataset. Equation (3) can then be rewritten as:

$$p(\boldsymbol{\theta},\boldsymbol{\mu}_\theta,\boldsymbol{\Sigma}_\theta \mid \boldsymbol{D}) \propto p(\boldsymbol{\mu}_{\boldsymbol{\theta}},\boldsymbol{\Sigma}_{\boldsymbol{\theta}}) \prod_{i=1}^{N_D} N(\boldsymbol{\theta}_i \mid \boldsymbol{\mu}_\theta,\boldsymbol{\Sigma}_\theta) N(\boldsymbol{\theta}_i|\boldsymbol{\theta}_i^*,\boldsymbol{\Sigma}_{\boldsymbol{\theta}_i}^*). \tag{10}$$

Subsequently, the posterior distributions can be solved as:

$$\begin{aligned} p(\boldsymbol{\mu}_\theta,\boldsymbol{\Sigma}_\theta \mid \boldsymbol{D}) \propto p(\boldsymbol{\mu}_{\boldsymbol{\theta}},\boldsymbol{\Sigma}_{\boldsymbol{\theta}}) \prod_{i=1}^{N_D} \int & N(\boldsymbol{\theta}_i \mid \boldsymbol{\mu}_\theta,\boldsymbol{\Sigma}_\theta) \\ & \times N\left(\boldsymbol{\theta}_i|\boldsymbol{\theta}_i^*,\boldsymbol{\Sigma}_{\boldsymbol{\theta}_i}^*\right) \mathrm{d}\boldsymbol{\theta}_i. \end{aligned} \tag{11}$$

The integral in equation (11) can be solved by using the rule given in [42]:

$$\int N\left(\boldsymbol{\theta}_i|\boldsymbol{\mu}_{\boldsymbol{\theta}},\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\right) N\left(\boldsymbol{\theta}_i|\boldsymbol{\theta}_i^*,\boldsymbol{\Sigma}_{\boldsymbol{\theta}_i}^*\right) \mathrm{d}\boldsymbol{\theta}_i = N\left(\boldsymbol{\mu}_{\boldsymbol{\theta}}|\boldsymbol{\theta}_i^*,\boldsymbol{\Sigma}_{\boldsymbol{\theta}}+\boldsymbol{\Sigma}_{\boldsymbol{\theta}_i}^*\right). \tag{12}$$

Therefore, the posterior distributions of the hyper parameters are computed in the following form:

$$p(\boldsymbol{\mu}_\theta,\boldsymbol{\Sigma}_\theta \mid \boldsymbol{D}) \propto p(\boldsymbol{\mu}_{\boldsymbol{\theta}},\boldsymbol{\Sigma}_{\boldsymbol{\theta}}) \prod_{i=1}^{N_D} N\left(\boldsymbol{\mu}_\theta \mid \boldsymbol{\theta}_i^*,\boldsymbol{\Sigma}_\theta+\boldsymbol{\Sigma}_{\boldsymbol{\theta}_i}^*\right). \tag{13}$$

It is noted that analytical solution for the posterior distributions of the hyper parameters is achieved as shown in equation (13). It provides computational efficiency, as it eliminates the need for time-consuming methods like Markov Chain Monte Carlo (MCMC) or any other samplings. Also, it is beneficial for reliability updating as it allows for fast updates to reliability assessments without the need for time-intensive sampling.

To compute the reliability index, a scalar performance function is typically required. One common way is to define a

scalar limit state function based on a critical component or direction in the parameter space:

$$G(\boldsymbol{\theta}) = b - \boldsymbol{c}^T\boldsymbol{A}^T\boldsymbol{\theta} = b - (\boldsymbol{Ac})^T\boldsymbol{\theta} \tag{14}$$

where b is the threshold. It plays a key role in the limit state function $G(\boldsymbol{\theta})$, which defines whether the system is in a safe state ($G(\boldsymbol{\theta}) > 0$) or a failure state ($G(\boldsymbol{\theta}) < 0$). $\boldsymbol{c}$ is a weighting vector (or a direction vector) and $\boldsymbol{Ac}$ acts as a transformed system matrix. In reliability analysis, $\boldsymbol{c}$ may represent the direction of the parameter that has the greatest impact on system performance. The failure probability conditional on the samples $\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}$ and $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}$ is given by:

$$\begin{aligned} F^{(m)}\left(\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}\right) &= \int\limits_{G(\boldsymbol{\theta})\leqslant 0} N\left(\boldsymbol{\theta}|\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}\right) \mathrm{d}\boldsymbol{\theta} \\ &= \Phi\left(-\beta^{(m)}\right) \end{aligned} \tag{15}$$

where $\beta^{(m)}$ is the reliability index condition on a hyper parameter sample

$$\beta^{(m)} = \frac{\left(b - (\boldsymbol{Ac})^T\right)\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}}{\sqrt{(\boldsymbol{Ac})^T\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}(\boldsymbol{Ac})}}. \tag{16}$$

Therefore, the posterior probability of failure given multiple datasets is

$$\Pr(F|\boldsymbol{D}) = \frac{1}{N_s}\sum_{m=1}^{N_s}\Phi\left(-\beta^{(m)}\right) \tag{17}$$

which is the average of the failure probabilities over all hyper parameter samples.

3.2. Structural dynamical model

This second case applies the HBM framework to a dynamical structural model, which differs from the previous linear case. Unlike linear models, where an analytical solution for the posterior distribution is achievable, dynamic models are more complex, and an analytical solution is not feasible. While several studies have used asymptotic approximations [42, 50] or variational inference [58] to obtain semi-analytical solutions for such systems, this work employs a two-stage sampling approach to ensure the accuracy of the HBM framework [59, 60]. The two-stage approach helps capture the full posterior distribution of the hyper parameters, ensuring accurate uncertainty quantification and reliability updating using multiple datasets.

Consider a dynamical structural model which describes the behavior of a structural system under varying conditions, and let $\boldsymbol{\theta} \in R^{N_\theta}$ be the set of material/structural model parameters to be estimated using data, and $N_{\boldsymbol{\theta}}$ is the total number of the unknown parameters in the set $\boldsymbol{\theta}$. The data used for this model can take the form of modal properties, such as natural frequencies, mode shapes, and damping ratios, or time histories, including displacement, velocity, acceleration or strain data over time. These data types are essential for capturing the dynamic response of the structure and are incorporated into the HBM framework for parameter estimation and uncertainty quantification.

Let the dataset be defined as $\boldsymbol{D} = \{\boldsymbol{D}_i;\ i = 1, 2, \ldots, N_D\}$, where N_D is the number of datasets which is the same with the first case. Followed by equation (3), the posterior distribution of the hyper parameters $p(\boldsymbol{\psi}|\boldsymbol{D})$ can be further computed by marginalization of the distribution $p(\boldsymbol{\theta}, \boldsymbol{\psi}|\boldsymbol{D})$ over the model parameters $\boldsymbol{\theta}$:

$$p(\boldsymbol{\psi}|\boldsymbol{D}) \propto p(\boldsymbol{\psi}) \prod_{i=1}^{N_D} \int p(\boldsymbol{\theta}_i|\boldsymbol{\psi}) p(\boldsymbol{D}_i|\boldsymbol{\theta}_i)\, \mathrm{d}\boldsymbol{\theta}_i. \tag{18}$$

The integral in equation (18) can be solved using MC simulation, given by:

$$\int p(\boldsymbol{\theta}_i|\boldsymbol{\psi}) p(\boldsymbol{D}_i|\boldsymbol{\theta}_i)\, \mathrm{d}\boldsymbol{\theta}_i \approx \frac{1}{N_s}\sum_{l=1}^{N_s} p\left(\boldsymbol{\theta}_i^{(l)}\middle|\boldsymbol{\psi}\right) \tag{19}$$

where $\boldsymbol{\theta}_i^{(l)}$ is the lth sample taken from the likelihood function $p(\boldsymbol{D}_i|\boldsymbol{\theta}_i)$ for the ith dataset, which can be obtained using the Transitional MCMC sampling algorithm [29, 60]. N_s is the number of samples for the model parameters. Substituting equation (19) into equation (18) leads to the following form:

$$p(\boldsymbol{\psi}|\boldsymbol{D}) \propto p(\boldsymbol{\psi}) \prod_{i=1}^{N_D} \frac{1}{N_s}\sum_{l=1}^{N_s} p\left(\theta_i^{(l)}|\boldsymbol{\psi}\right). \tag{20}$$

By replacing the hyper parameter $\boldsymbol{\psi}$ with hyper mean and hyper covariance matrix, equation (20) takes the form:

$$p(\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}|\boldsymbol{D}) \propto p(\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) \prod_{i=1}^{N_D} \frac{1}{N_s}\sum_{l=1}^{N_s} N\left(\boldsymbol{\theta}_i^{(l)}\middle|\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}\right). \tag{21}$$

Equation (21) can be then solved by a two-stage sampling approach. In the first stage, the sample $\boldsymbol{\theta}_i^{(l)}$ needs to be generated from the distribution $p(\boldsymbol{D}_i|\boldsymbol{\theta}_i)$. It thus requires the model runs which is the most expensive computational part. In the second step, it does not require the model runs and only the samples from the first step need to be incorporated into equation (21). The samples in the second step are also generated using the Transitional MCMC sampling algorithm [29, 60]. Once the samples of the hyper parameters are available, reliability could be conditional on these samples.

In this study, two cases are considered for reliability updating where the first case only accounts for the means of the hyper parameters while the second case considers all the samples of the hyper parameters. The updates of the reliability between two cases are expected to be close when given sufficient number of datasets. This is because the uncertainty

of the hyper parameters would decrease as the number of datasets increases [58]. Given a large number of datasets, the posterior distribution would be a peaked distribution for which their uncertainty does not play a role for reliability updating. This scenario can be further demonstrated by the insights given in [58]. in the sense of updating reliability.

Considering only the means of the hyper parameters, the posterior probability of failure given multiple datasets is expressed as:

$$\begin{aligned}\Pr(F|\boldsymbol{D}) &= \int_{\substack{\boldsymbol{\theta}\in\mathbb{R}^{N_{\boldsymbol{\theta}}}\\ \boldsymbol{\varphi}\in\mathbb{R}^{N_{\boldsymbol{\varphi}}}}} p(\boldsymbol{\varphi})\,\mathrm{I}_F(\boldsymbol{\theta},\boldsymbol{\varphi})\,p(\boldsymbol{\theta}|\boldsymbol{D})\,\mathrm{d}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{\theta}\\ &= \int_{\substack{\boldsymbol{\theta}\in\mathbb{R}^{N_{\boldsymbol{\theta}}}\\ \boldsymbol{\varphi}\in\mathbb{R}^{N_{\boldsymbol{\varphi}}}}} p(\boldsymbol{\varphi})\,\mathrm{I}_F(\boldsymbol{\theta},\boldsymbol{\varphi})\,N\left(\boldsymbol{\theta}|\overline{\boldsymbol{\mu}}_{\boldsymbol{\theta}},\overline{\boldsymbol{\Sigma}}_{\boldsymbol{\theta}}\right)\mathrm{d}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{\theta}\end{aligned} \tag{22}$$

where $\bar{\boldsymbol{\mu}}_{\boldsymbol{\theta}}$ and $\bar{\boldsymbol{\Sigma}}_{\boldsymbol{\theta}}$ are the means of the hyper parameters, $I_{\mathrm{F}}(\boldsymbol{\theta},\boldsymbol{\varphi})$ is the indicator function such that $I_{\mathrm{F}}(\boldsymbol{\theta},\boldsymbol{\varphi})=1$ when the limit state function satisfies $G(\boldsymbol{\theta},\boldsymbol{\varphi})\leqslant 0$, otherwise it will be zero. Note that $\boldsymbol{\varphi}$ is the set of parameters associated with other uncertainties (e.g. input related uncertainties) that are not included in the parameter set $\boldsymbol{\theta}$ to be inferred by the HBM framework. It is assumed that $\boldsymbol{\varphi}$ is uncorrelated to the model parameters and a PDF $p(\boldsymbol{\varphi})$ is assigned to quantify uncertainties in $\boldsymbol{\varphi}$. Subset simulation is used in this study to compute the probability of failure using multiple datasets.

When all the samples of the hyper parameters are considered, the posterior probability of failure given multiple datasets is given by:

$$\begin{aligned}\Pr(F|\boldsymbol{D}) &= \int_{\substack{\boldsymbol{\theta}\in\mathbb{R}^{N_{\boldsymbol{\theta}}}\\ \boldsymbol{\varphi}\in\mathbb{R}^{N_{\boldsymbol{\varphi}}}}} p(\boldsymbol{\varphi})\,\mathrm{I}_F(\boldsymbol{\theta},\boldsymbol{\varphi})\,p(\boldsymbol{\theta}|\boldsymbol{D})\,\mathrm{d}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{\theta}\\ &= \frac{1}{N_s}\sum_{m=1}^{N_s}\int_{\substack{\boldsymbol{\theta}\in\mathbb{R}^{N_{\boldsymbol{\theta}}}\\ \boldsymbol{\varphi}\in\mathbb{R}^{N_{\boldsymbol{\varphi}}}}} p(\boldsymbol{\varphi})\,\mathrm{I}_F(\boldsymbol{\theta},\boldsymbol{\varphi})\,N\left(\boldsymbol{\theta}|\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)},\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}\right)\mathrm{d}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{\theta}\\ &= \frac{1}{N_s}\sum_{m=1}^{N_s} F^{(m)}\left(\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)},\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}\right)\end{aligned} \tag{23}$$

where $F^{(m)}(\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)},\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)})$ is the failure probability conditional on the samples of the hyper parameters $\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}$ and $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}$, given by:

$$\begin{aligned}F^{(m)}\left(\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)},\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}\right) = \int_{\substack{\boldsymbol{\theta}\in\mathbb{R}^{N_{\boldsymbol{\theta}}}\\ \boldsymbol{\varphi}\in\mathbb{R}^{N_{\boldsymbol{\varphi}}}}} & p(\boldsymbol{\varphi})\,I_F(\boldsymbol{\theta},\boldsymbol{\varphi})\\ & \times N\left(\boldsymbol{\theta}|\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)},\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}\right)\mathrm{d}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{\theta}\end{aligned} \tag{24}$$

in which the subset simulation could be implemented to compute $F^{(m)}(\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)},\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)})$. The model update and reliability analysis process of HBM method based on two-stage sampling method is presented, as shown in figure 1. It is worth noting that the method ensures the accuracy of the framework by sampling calculation in two stages. Detailed information for the sampling setup is given in the Illustrative examples.

4. Illustrative examples

4.1. HBM framework for a linear mathematical model

The linear model of equation (4) is taken as an example to explore the uncertainty and reliability updating using data. The model is updated by three parameters $\boldsymbol{\theta}=[\theta_1,\theta_2,\theta_3]^T$ with nominal values equaling to one. The coefficient matrix $\boldsymbol{A}^T$ is generated by random numbers from one to five and has dimensions $N_d\times 3$, where N_d is the number of data points in a dataset. For exploring how the number of data points would affect the parameter uncertainty in a dataset, N_d is selected as 50, 100, 200 and 500, respectively. The dataset is simulated based on the Gaussian distribution of the hyper mean $\boldsymbol{\mu}_{\boldsymbol{\theta}}=[1,1,1]^T$ and hyper covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}=\mathrm{diag}\left(0.05^2,0.05^2,0.05^2\right)$. This simulation is consistent with the hierarchical Bayesian framework, where model parameters are modeled using a hierarchical structure with Gaussian prior distributions characterized by hyperparameters. Subsequently, 2% Gaussian noise is added to the measurements. Then the mean $\boldsymbol{\theta}^*$ and the covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^*$ are solved according to equation (8). Table 1 shows the changes of standard deviation $(\sigma_i,\ i=1,2,3)$ in the covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^*$ corresponding to each parameter when N_d takes different values. It is found that the value of standard deviation $(\sigma_i,\ i=1,2,3)$ decreases with the increase of data points, indicating that the epistemic uncertainty decreases with the increase of data points.

Once the mean $\boldsymbol{\theta}^*$ and the covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^*$ are obtained for each dataset, the distribution of the hyper parameters could be computed using equation (13). In this example, 5000 samples were used for analysis based on the HBM method, and the data points in a dataset are chosen as 500. Figure 2 depicts the hyper parameter posterior distribution of model parameters for 500 datasets, with μ_θ and σ_θ representing the hyper parameters associated with the model parameters. It is worth noting that the dimension of the coefficient matrix $\boldsymbol{A}^T$ is 500×3, that is, the number of data points in each dataset is 500. It can be seen that the mean values of the hyper parameters are in good agreement with their nominal values identified using the 500 datasets.

In order to explore the effects of different datasets on the posterior probability distribution, different data sets are also simulated. The model is updated using four datasets of varying numbers from $N_{\mathrm{D}}=50$ to $N_{\mathrm{D}}=500$, where N_{D} defines the number of datasets. The results in table 2 show the statistics of the posteriori distribution of hyper parameters obtained based on HBM method for four datasets. As can be seen from table 2, the mean values of hyper mean and hyper standard deviation obtained by using different data sets based on HBM approach are close to the nominal values. It can also be seen that when given a smaller data set ($N_{\mathrm{D}}=50$), the mean of the posterior sample deviates from the nominal value of the hyper parameter. In this case, the hyper parameters have great uncertainty. However, with the increase of the number of data sets, the mean value of the hyper parameter posterior sample is closer to the nominal value of the hyper parameter, and the uncertainty of the hyper parameter decreases. The above results confirm that the uncertainty of hyper parameters decreases

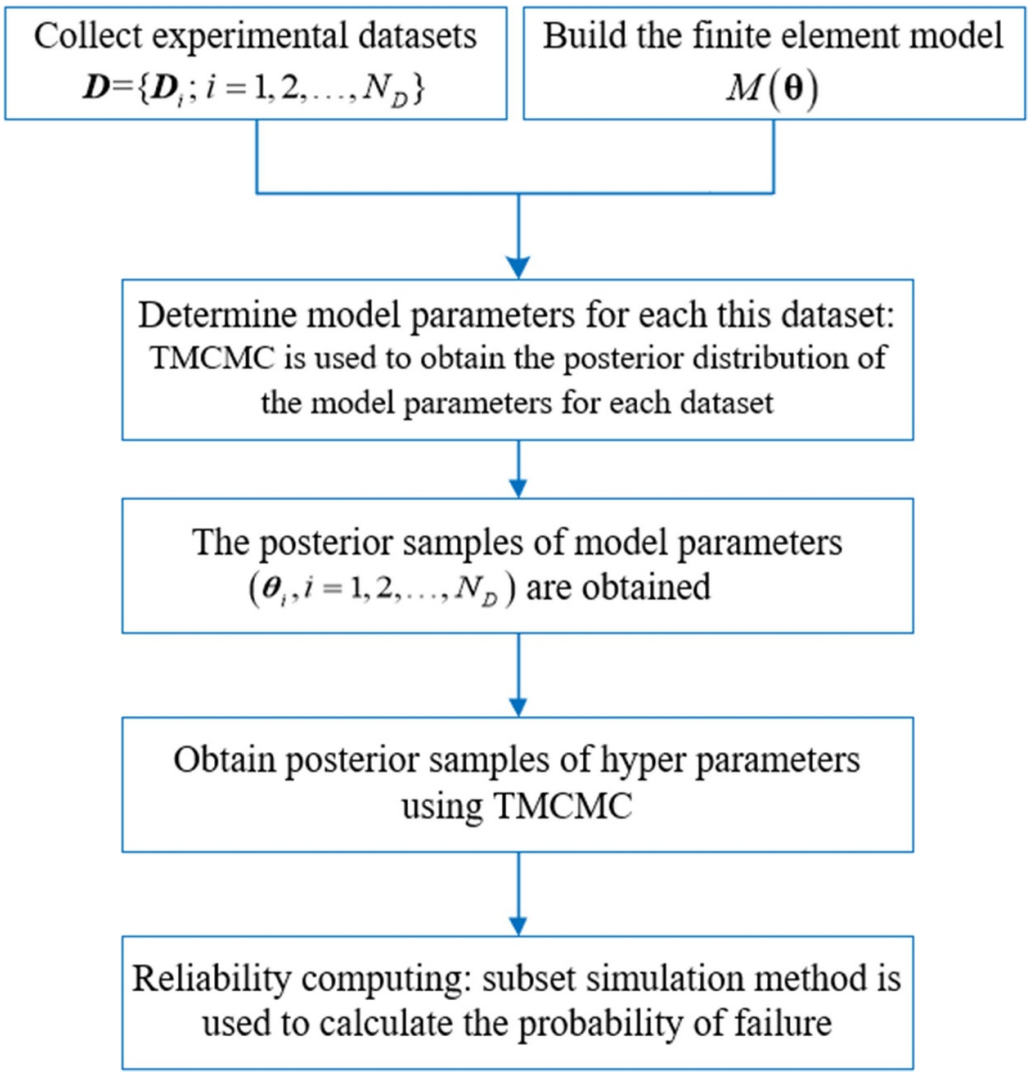


Figure 1. The diagram of two-stage sampling method.

Table 1. Standard deviation at different data points.

	$N_d = 50$	$N_d = 100$	$N_d = 200$	$N_d = 500$
$\sigma_1\left(10^{-4}\right)$	5.3115	5.2618	5.1702	5.0436
$\sigma_2\left(10^{-4}\right)$	5.4899	5.2377	5.1923	5.1912
$\sigma_3\left(10^{-4}\right)$	5.3262	5.3062	5.1502	5.0858

with the increase of the number of datasets. When given sufficient number of datasets, the mean of the hyper parameter will be closer to its nominal value. At the same time, the sampling method is used to calculate the posterior distribution of the same data set, and the results are shown in table 3. Through the comparison between tables 2 and 3, it is found that the slight difference between the two is observed when using the small number of datasets, while similar results are achieved when given sufficient number of the datasets. In addition, figure 3 shows the PDF of hyper parameter samples of different data sets. From the results in figure 3, it can be seen that as the data set increases, the probability distribution of the hyper parameter samples takes on a narrower range of values i.e. the distribution of the corresponding samples is denser, and the uncertainty is smaller. The more data sets, the closer the hyper mean and hyper standard differences are to the nominal values. However, regardless of the size of the data set, the nominal value of the hyper parameter is included in this uncertainty. The above content is the same as the results obtained through the analysis in table 2, which verifies the property that the more data sets, the smaller the uncertainty of the hyper parameters.

The posterior distribution of the model parameters can then be obtained with the samples of their hyper parameters. The results of the model parameters using the proposed HBM framework are compared with those from the classical Bayesian modeling (CBM) framework. For CBM, all the

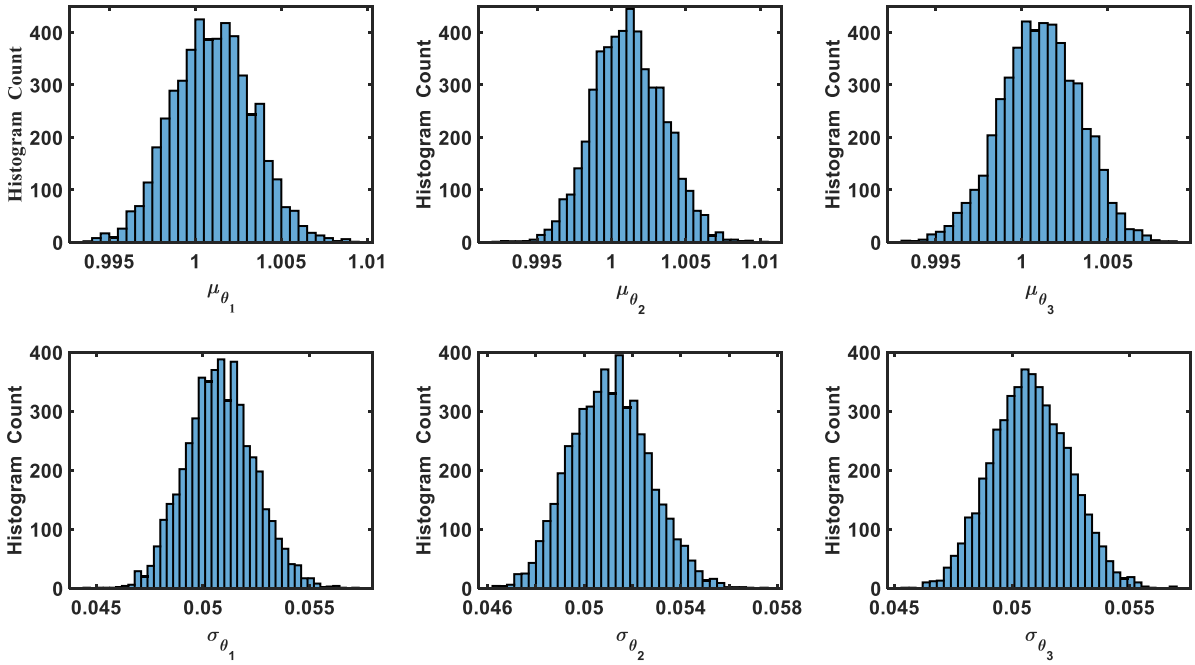


Figure 2. Posterior distribution of the hyper parameters using HBM method ($N_{\mathrm{D}} = 500$).

Table 2. Statistics of hyper parameters based on HBM method.

		θ_1		θ_2		θ_3	
		μ_{θ_1}	σ_{θ_1}	μ_{θ_2}	σ_{θ_2}	μ_{θ_3}	σ_{θ_3}
$N_{\mathrm{D}} = 50$	Mean value	0.9975	0.0565	0.9969	0.0557	0.9977	0.0569
	Standard deviation	0.0076	0.0057	0.0076	0.0057	0.0079	0.0061
$N_{\mathrm{D}} = 100$	Mean value	1.0031	0.0489	1.0034	0.0489	1.0034	0.0486
	Standard deviation	0.0048	0.0034	0.0050	0.0034	0.0048	0.0036
$N_{\mathrm{D}} = 200$	Mean value	1.0023	0.0516	1.0025	0.0518	1.0027	0.0516
	Standard deviation	0.0035	0.0025	0.0036	0.0025	0.0037	0.0025
$N_{\mathrm{D}} = 500$	Mean value	1.0010	0.0508	1.0011	0.0511	1.0010	0.0508
	Standard deviation	0.0023	0.0016	0.0023	0.0016	0.0023	0.0016

Table 3. Hyper parameter statistics of HBM method based on sampling.

		θ_1		θ_2		θ_3	
		μ_{θ_1}	σ_{θ_1}	μ_{θ_2}	σ_{θ_2}	μ_{θ_3}	σ_{θ_3}
$N_{\mathrm{D}} = 50$	Mean value	0.9988	0.0468	0.9954	0.0473	0.9998	0.0463
	Standard deviation	0.0066	0.0050	0.0064	0.0048	0.0067	0.0045
$N_{\mathrm{D}} = 100$	Mean value	0.9960	0.0530	0.9984	0.0526	0.9993	0.0524
	Standard deviation	0.0060	0.0040	0.0056	0.0041	0.0057	0.0041
$N_{\mathrm{D}} = 200$	Mean value	0.9989	0.0505	0.9990	0.0511	0.9981	0.0503
	Standard deviation	0.0035	0.0025	0.0036	0.0025	0.0035	0.0026
$N_{\mathrm{D}} = 500$	Mean value	0.9996	0.0502	0.9998	0.0501	0.9997	0.0507
	Standard deviation	0.0022	0.0015	0.0021	0.0015	0.0021	0.0015

datasets are incorporated in a dataset, and 5000 samples are generated using the TMCMC algorithm for the posterior distribution of the model parameters. The results of the posterior distribution of the model parameters are given in figure 4. Good agreements are observed for the mean values of the model parameters between two approaches, while the CBM method provides an extremely thin uncertainty bound which cannot capture the uncertainty due to variability from dataset to dataset. The reason for this result is that the CBM method seriously underestimates the uncertainty of the parameters, so the probability density curve of the model parameters obtained by CBM method is narrow.

Reliability analysis is then conducted using the hyper samples given in figure 2. The limit state function is given by $G(\boldsymbol{\theta}) = b - (\boldsymbol{Ac})^T\boldsymbol{\theta}$ with $\boldsymbol{c}^T$ equal to all-one matrix with dimension 1×500. Figure 5 shows the failure probability

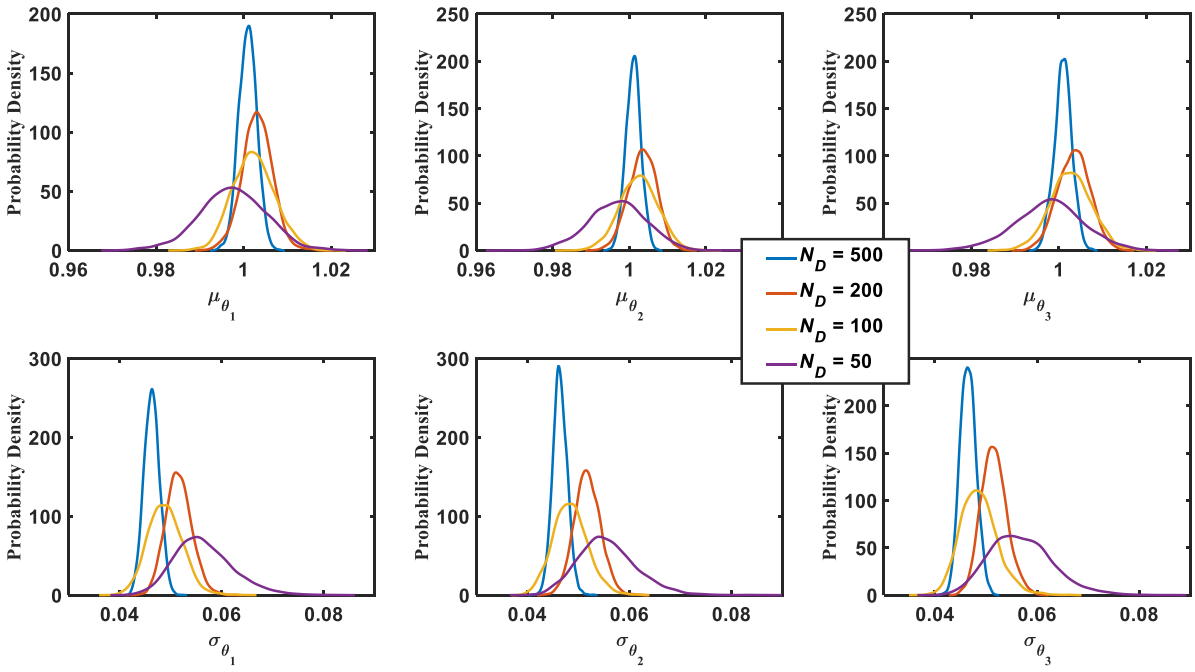


Figure 3. Distributions of hyper parameters corresponding to different datasets.

based on the exceedance level b using all samples of hyper parameters obtained based on HBM method and the samples of the model parameters using CBM method. The failure probability given by the two methods differs greatly, and the values of failure probability differ by many orders of magnitude under the same vertical coordinate. The above situation is caused by different uncertainties, that is, different uncertainties lead to different reliability between two approaches. The large difference between the two in the figure is mainly because the CBM method seriously underestimates the uncertainty.

4.2. Dynamical system: 3 degrees-of-freedom (DOF) spring mass model

Taking the three DOF spring mass model in structural dynamics as an example, the performance of the proposed method is evaluated. The structure is shown in figure 6, where the system is fixed to the left. The nominal mass (m_0^i) is 1 kg and the stiffness (k_0^i) per chain is 1800 N m^{-1}. The damping ratio of each mode is assumed to be 0.02. The parameter to be updated is the stiffness of each spring of the system. Three model parameters $\boldsymbol{\theta} = [\theta_1, \theta_2, \theta_3]^T$ are used to parameterize the model stiffness, defined as $k_i = k_0^i \theta_i$, and representing the product of nominal stiffness and model parameters. Similar with the hierarchical Bayesian framework, a Gaussian distribution with hyper mean $\mu_{\boldsymbol{\theta}} = [1,1,1]^T$ and hyper covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\theta}} = \text{diag}\left(0.05^2, 0.05^2, 0.05^2\right)$ is used to simulate the measured dataset. The acceleration measurements are generated as the response for further updating purpose. Sensors are located on all three DOFs so that all the measurements can be considered for parameter estimation. Each dataset D_i is contaminated with 2% Gaussian white noise in order to add measurement error.

The two-stage algorithm introduced in section 3.2 is used to compute the posterior distribution of the hyper parameters. The number of the samples of the hyper parameters is specified as 5000. The covariance tolerance for weight normalization is set to 1 by default, while parallel computing is used to improve computational efficiency. The algorithm stops when it reaches the tolerance values. Four cases with different number of datasets ranging from $N_D = 10$ to $N_D = 500$ are explored to compute the posterior distribution of the hyper parameters. In particular, the means along with the standard deviations of the hyper parameters are reported in table 4. It can be observed that the mean values of the hyper mean and hyper standard deviation of the model parameters and prediction errors obtained from different data sets are close to the nominal values. However, when given a smaller data set ($N_D = 10$), the mean of the posterior sample deviates from the nominal value of the hyper parameter. Therefore, the hyper parameters have large uncertainty. With the increase of the number of data sets, the posterior sample mean is closer to the nominal value of the hyper parameter, and the uncertainty of the hyper parameter decreases. The above results can prove that with the increase of the data set, the uncertainty row of the hyper parameter decreases, and when the data set is large enough, the mean value of the hyper parameter will be closer to its true value. In addition, the PDF of the hyper parameter posterior sample corresponding to the model parameters of different datasets is shown in figure 7. It can be observed that when a small dataset ($N_D = 10$) is used, the mean of the posterior sample deviates significantly from the nominal value of the hyper parameter, and the uncertainty associated with the hyper parameter is relatively large. However, the nominal value of the hyper parameter still lies within the range of this uncertainty. As the size of the dataset increases, the uncertainty of the hyper

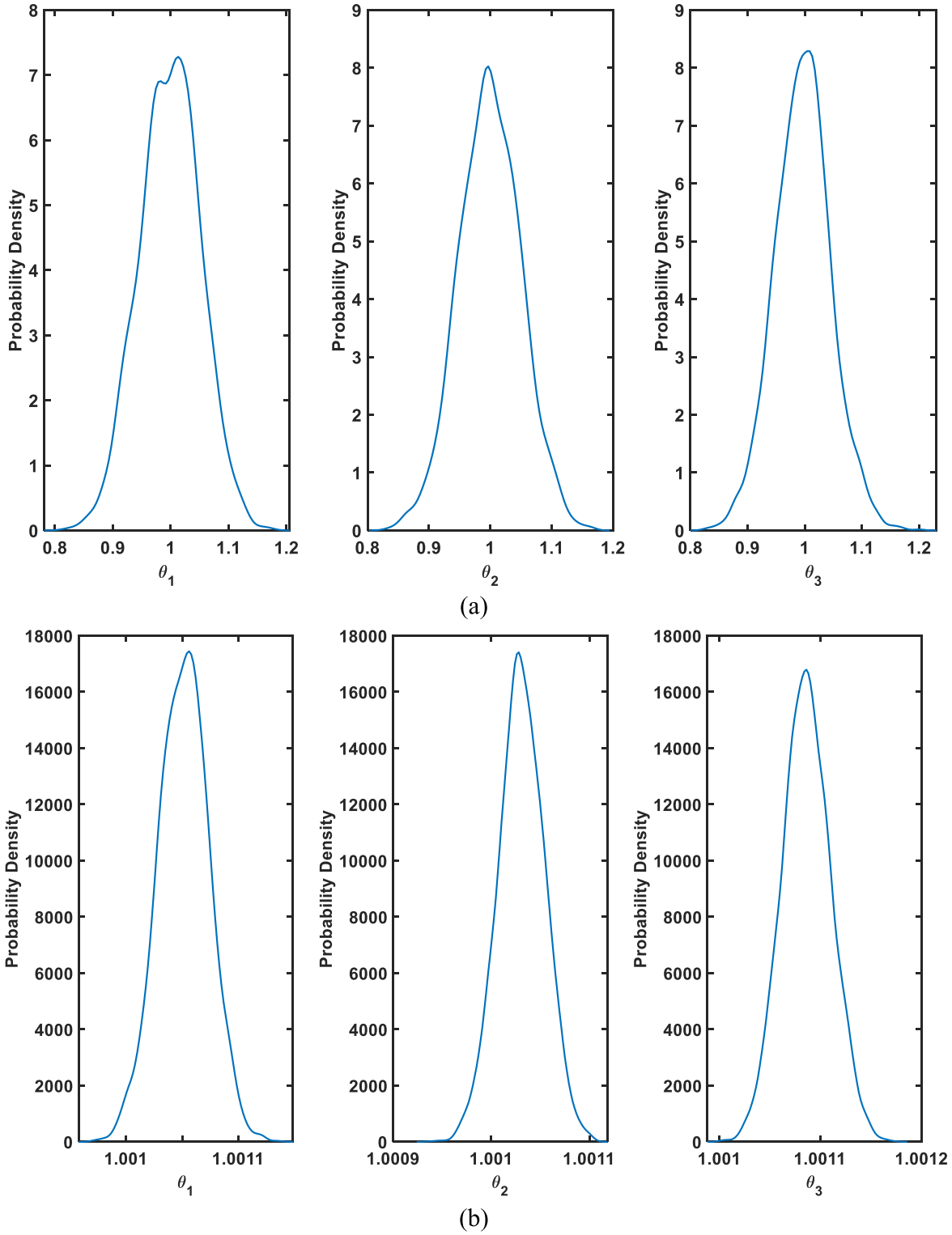


Figure 4. Model parameter distribution using (a) HBM method (b) CBM method.

parameters decreases, and the sample mean of most parameters approaches their nominal values. This trend is consistent with the results presented in table 4. An increase in the number of datasets leads to a narrower range for the probability density function, meaning the distribution of the samples becomes denser, thereby reducing uncertainty. Figure 8 further illustrates the posterior distributions of the hyper parameters and prediction error variance for case $N_D = 500$. These posterior samples will subsequently be used for reliability updating purposes. Note that in this case limitations also appear when using

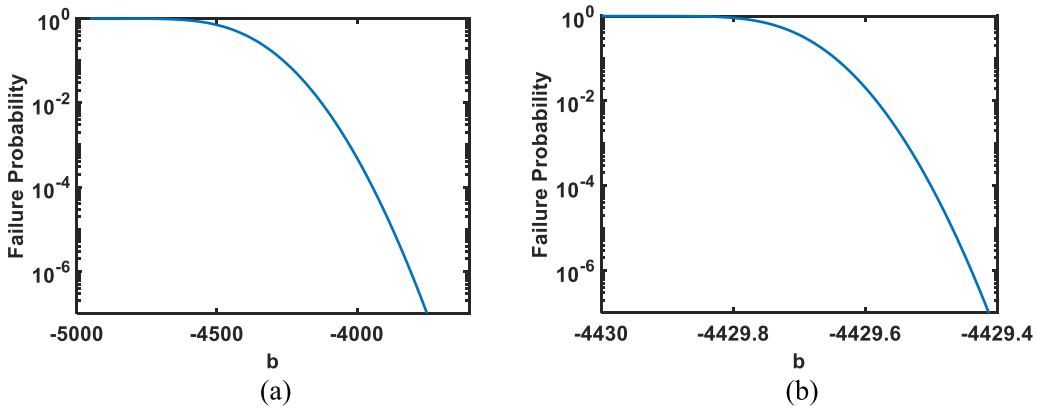


Figure 5. Reliability results using (a) HBM (b) CBM.

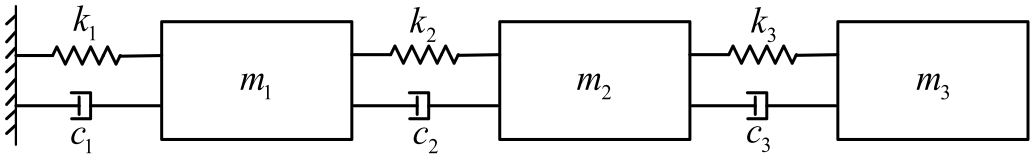


Figure 6. 3-DOF mass spring system.

Table 4. Estimate the mean value of the hyper parameters using the HBM method.

		θ_1		θ_2		θ_3		σ	
		μ_{θ_1}	σ_{θ_1}	μ_{θ_2}	σ_{θ_2}	μ_{θ_3}	σ_{θ_3}	μ_σ	σ_σ
$N_D = 10$	Mean value	1.0179	0.0606	1.0354	0.0542	1.0232	0.0584	0.0202	0.0002
	Standard deviation	0.0205	0.0122	0.0216	0.0122	0.0198	0.0142	1.2570×10^{-4}	1.3321×10^{-4}
$N_D = 50$	Mean value	0.9896	0.0410	0.9919	0.0432	0.9923	0.0415	0.0199	0.0002
	Standard deviation	0.0058	0.0043	0.0061	0.0050	0.0058	0.0042	5.1974×10^{-5}	9.4092×10^{-5}
$N_D = 200$	Mean value	0.9935	0.0451	0.9933	0.0454	0.9933	0.0453	0.0200	0.0001
	Standard deviation	0.0032	0.0023	0.0032	0.0023	0.0031	0.0023	2.7965×10^{-5}	5.6624×10^{-5}
$N_D = 500$	Mean value	1.0032	0.0525	1.0035	0.0524	1.0035	0.0523	0.0200	0.0001
	Standard deviation	0.0024	0.0017	0.0023	0.0015	0.0023	0.0016	1.6281×10^{-5}	3.2013×10^{-5}

the two-stage sampling approach. When the number of model parameters is large, the two-stage sampling approach may produce biased results due to insufficient exploration of the high-dimensional parameter space. This is particularly problematic in complex dynamic systems where the interactions between parameters are nonlinear and highly coupled. As more samples are used to improve accuracy, the computational cost increases significantly, making the method less feasible for large-scale problems. Alternatives could be employed by using asymptotic approximation [57] or variational inference [58] of the posterior distribution of the hyper parameters.

With the samples of the hyper parameters from figure 8, the distribution of the model parameters could be obtained. Herein, two cases from the HBM framework are considered, where the first case only considers the mean of the hyper parameters with neglecting their uncertainties while the second case accounts for all the samples of the hyper parameters. These two cases are also compared with the result from CBM approach where all datasets are combined for parameter inference. Figure 9 shows the parameter distribution using the HBM and CBM approaches with HBM considering the two cases mentioned above. It is seen that the results from the two cases in the HBM framework are quite similar, primarily because the number of datasets is large, and the uncertainty of the hyper parameters does not play a significant role. In contrast, the CBM approach produces a highly peaked distribution, as it underestimates parameter uncertainties. This narrow distribution is unreasonable when accounting for parameter variability, as it fails to properly reflect the inherent uncertainty in the parameters.

In addition, the probability that the maximum displacement exceeds a predefined threshold is calculated. For this purpose, the limit state function is defined as $G(\boldsymbol{\theta}, \boldsymbol{\varphi}) = d_i(\boldsymbol{\theta}, \boldsymbol{\varphi}) - d_0$ under the constraint that the displacement of the system does not exceed d_0, where $d_i(\boldsymbol{\theta}, \boldsymbol{\varphi})$ is the displacement of the ith DOF, and d_0 is the threshold. The uncertainty of the input is also considered which is generated as a white noise sequence with 1000 parameters having a mean of zero and a standard deviation of one. Figure 10 presents three sets of results: two from the HBM framework and one from the CBM approach.

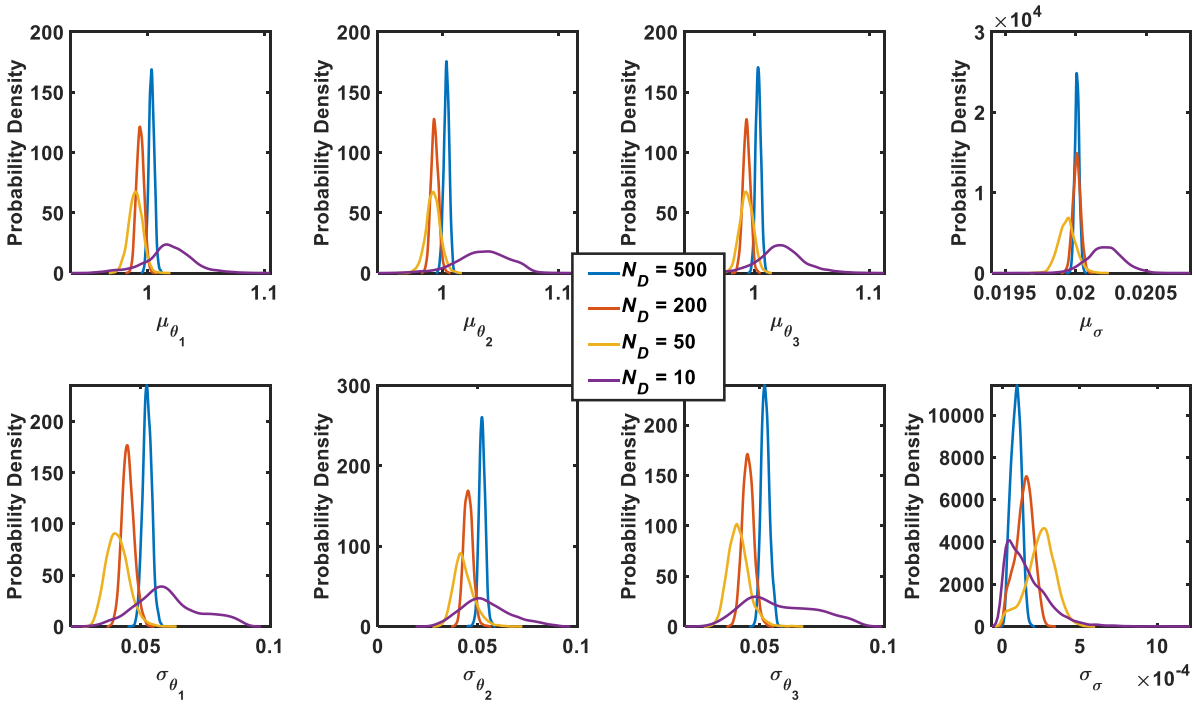


Figure 7. Distributions of the hyper parameters corresponding to $N_D = 10$, 50, 200 and 500.

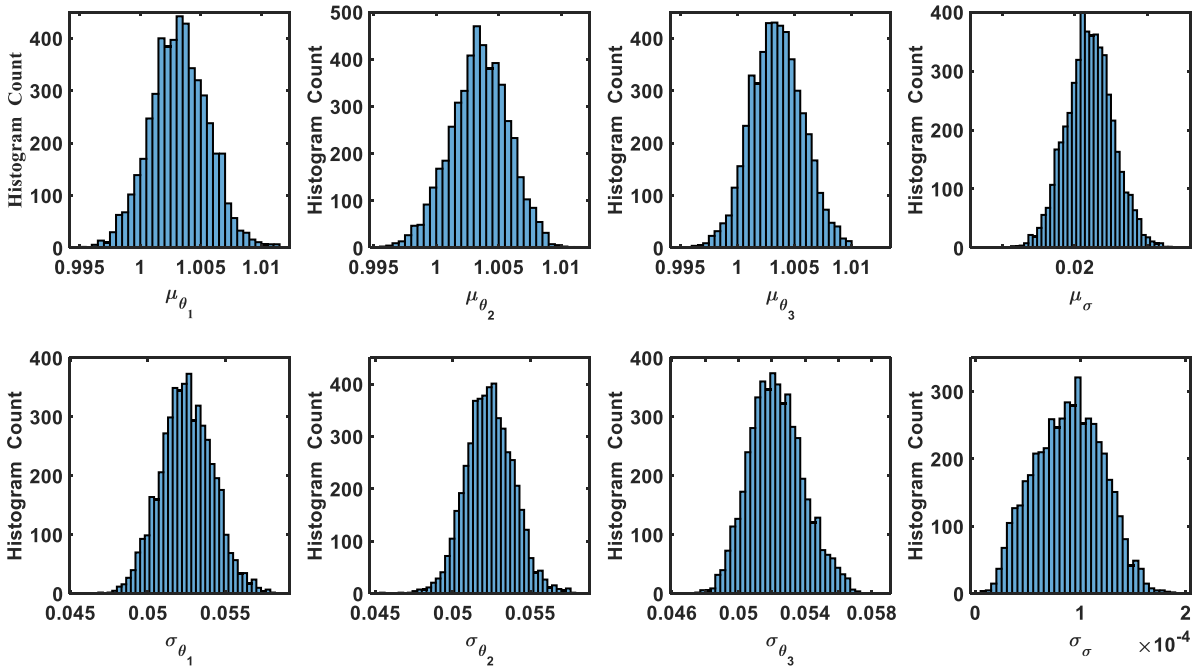


Figure 8. Posterior distribution of the hyper parameters using HBM method ($N_D = 500$).

In the HBM framework, the results are similar because a large number of datasets are used, leading to relatively peaked distributions of hyper parameters. In contrast, the CBM approach shows markedly different results, giving predictions that are essentially one to many orders of magnitude different compared to the first two. The reason for this difference is that the CBM method seriously underestimates the uncertainty. These differences are further illustrated in figure 11, which shows the predicted probability density function of the maximum displacement of the system. The CBM-based PDF is the narrowest, clearly indicating that CBM significantly underestimates uncertainty compared to the HBM framework. On the other hand, the PDF based on the HBM hyperparameter sample mean is consistent with the prediction obtained using all hyperparameter samples as sufficient number of datasets used in this case.

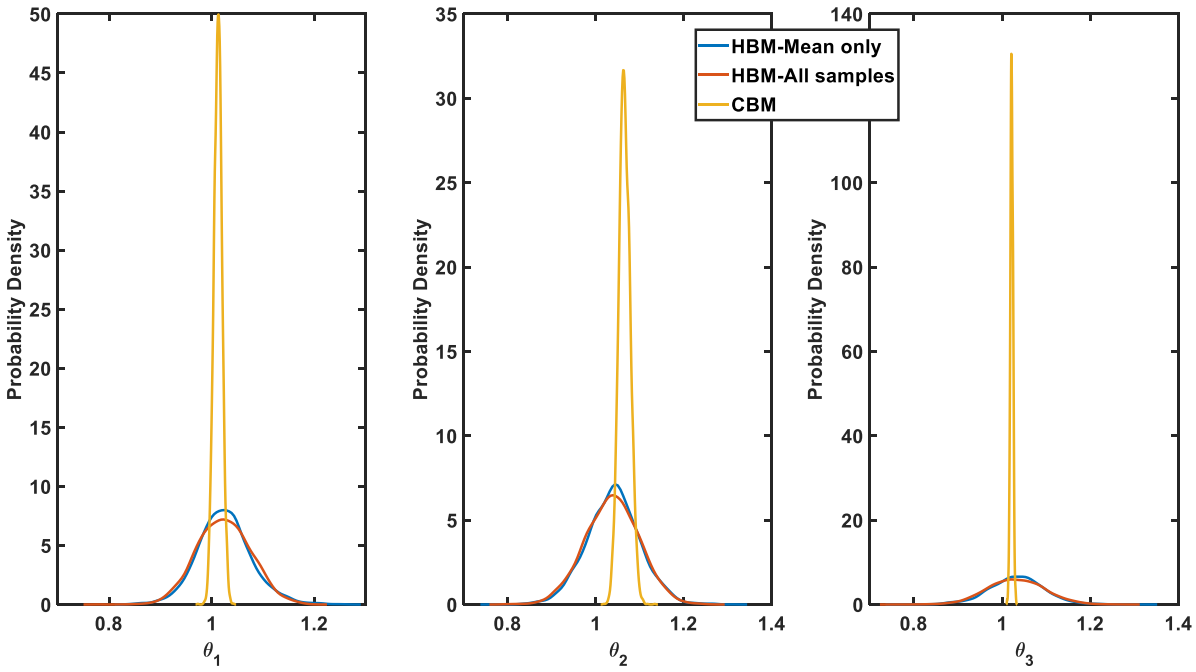


Figure 9. Model parameter distribution.

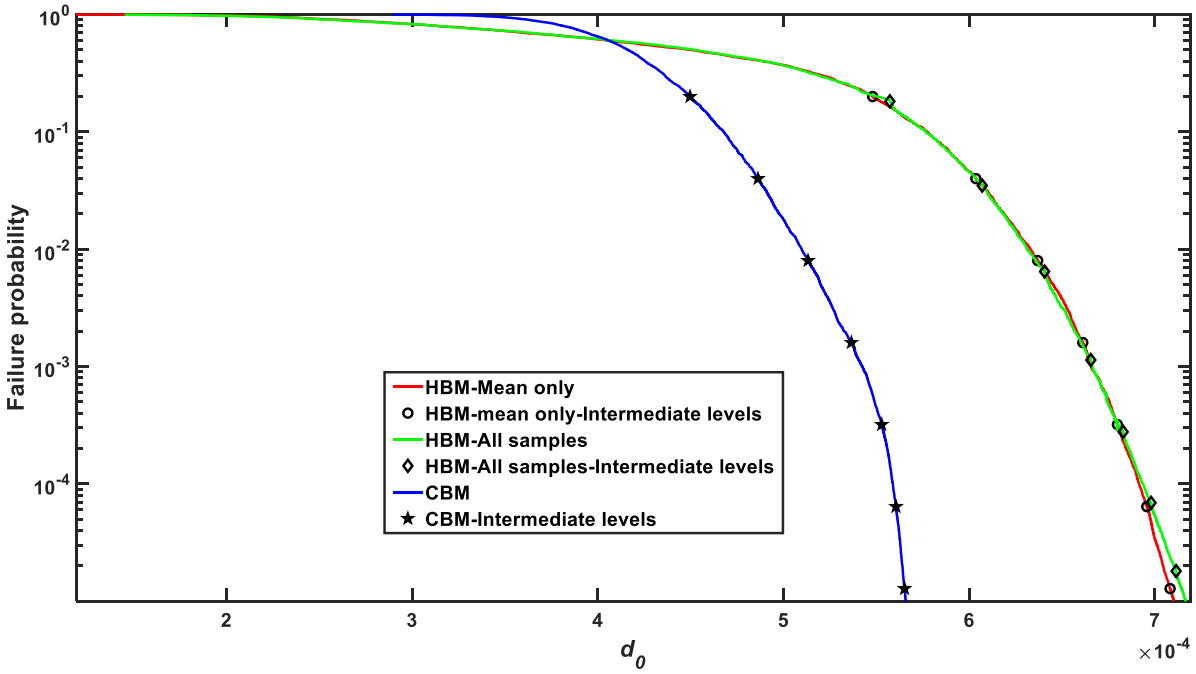


Figure 10. The failure probability of the maximum displacement.

5. Conclusion

This study developed a HBM framework for uncertainty quantification and reliability updating using data, with applications to a linear mathematical model and a dynamical structural model. For the linear model, an analytical solution was derived for the hyper parameters and reliability index, demonstrating the efficiency and precision of HBM in simpler cases. For the dynamical structural model, where analytical solutions are not feasible, a two-stage sampling approach was introduced to ensure the accuracy of the framework. The results highlight a critical difference between HBM and CBM in how uncertainty influences reliability predictions. In practical applications, the reliability analysis results obtained by the traditional Bayesian method may be too optimistic, which may lead to serious overestimation of safety. HBM maintains a realistic representation of uncertainty, resulting in reliability predictions that are robust and reflective of the true variability in

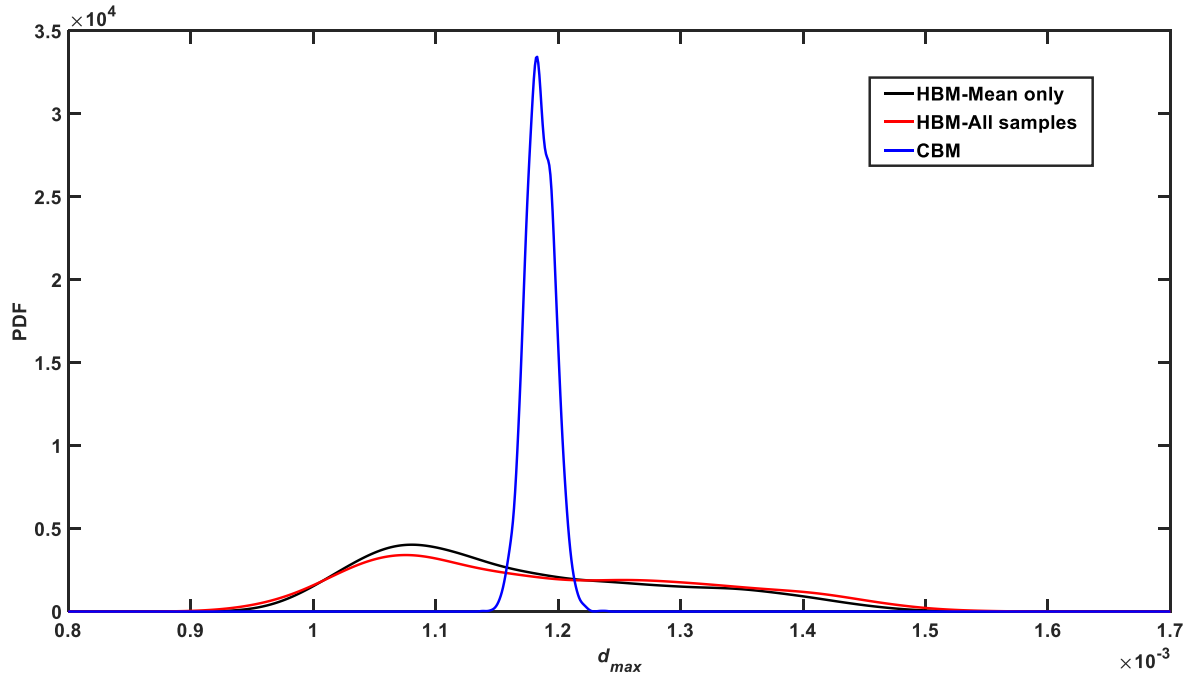


Figure 11. Posterior prediction distribution of maximum displacement.

the system. This work underscores the advantages of HBM for addressing challenges in reliability assessment and highlights its potential for further applications. Future work will focus on extending this framework to incorporate datasets collected over periodic intervals, allowing for continuous reliability updates. Additionally, efforts will be made to integrate damage detection into the framework, enabling more comprehensive monitoring and assessment of structural health. These advancements aim to enhance the capability of the HBM framework to address real-world engineering challenges, particularly in systems subject to evolving operational and environmental conditions.

The method presented in this paper can also be extended to multi-level models having different components or subsystems [61, 62] for reliability updating. The advantage of this method is that the analytical solutions of hyper parameters and reliability indexes of linear systems can be derived, so as to improve computational efficiency. However, the calculation cost of two-stage sampling method for nonlinear multi-level models increases significantly and may produce biased results. Potential solutions could be conducted using asymptotic approximation or variational inference of the hyperparameter posterior distribution.

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